

Article

The Hadamard product and recursively defined sequences

(to Margarita)

Sergei Dmitrievich Kazenas

Independent researcher, Moscow, Russia.; kzenas@pm.me

Received: 13 December 2019; Accepted: 27 January 2020; Published: 22 February 2020.

Abstract: In this paper, the approach to obtaining nontrivial formulas for some recursively defined sequences is illustrated. The most interesting result in the paper is the formula for the solution of quadratic map-like recurrence. Also, some formulas for the solutions of linear difference equations with variable coefficients are obtained. At the end of the paper, some integer sequences associated with a quadratic map are considered.

Keywords: The Hadamard product, recursively defined sequence, difference equation, quadratic map.

MSC: 39A10, 05A15.

1. Introduction

Let us begin with some notation. Let \mathbf{b} be a row vector and \mathbf{a} be a finite row vector; $(\mathbf{b})_j$ denotes j th element of vector \mathbf{b} , $|\mathbf{b}|$ denotes length of \mathbf{b} ; $\mathbf{a} \cup \mathbf{b} \stackrel{\text{def}}{=} ((\mathbf{a})_1, \dots, (\mathbf{a})_{|\mathbf{a}|}, (\mathbf{b})_1, \dots)$, $(\mathbf{b})_l^m \stackrel{\text{def}}{=} ((\mathbf{b})_j)_{j=l}^m$, $|\mathbf{b}|_x \stackrel{\text{def}}{=} \sum_{j=1}^{|\mathbf{b}|} (\mathbf{b})_j x^{j-1}$, $\mathbf{1}_m \stackrel{\text{def}}{=} (1)_{j=1}^m$, $\mathbf{0}_m \stackrel{\text{def}}{=} (0)_{j=1}^m$, $[l, m]_j \stackrel{\text{def}}{=} (1_{\infty} \times (\mathbf{0}_l \cup \mathbf{1}_m))_j$, where \times denotes the Kronecker product, $l \in \mathbb{N}$, $m \in \mathbb{N} \cup \{\infty\}$. Note that the last function can be expressed by the ceiling function: $[l, m]_j = \lceil (j+m)/(l+m) \rceil - \lceil j/(l+m) \rceil$.

Also, we use ordinary notation to denote the corresponding entrywise operations. For example, \mathbf{a}^2 expresses the Hadamard square: $\mathbf{a}^2 = ((\mathbf{a})_j^2)_{j=1}^{|\mathbf{a}|}$.

It should be noted that there are many papers on sequences generated by linear difference equations with variable coefficients. See, for example, [1–3]. The simple approach illustrated here involves constructing for each such sequence a corresponding recursive vector sequence, which can be explicitly expressed using the following property of Hadamard product: $(\mathbf{ac}) * (\mathbf{bd}) = (\mathbf{a} * \mathbf{b})(\mathbf{c} * \mathbf{d})$, where $|\mathbf{c}| = |\mathbf{a}|$, $|\mathbf{d}| = |\mathbf{b}|$ and $* \in \{\cup, \times\}$.

2. Linear recurrences

First we use this property with respect to concatenation.

Theorem 1. If x_1, x_2 are arbitrary numbers, a_n, b_n are arbitrary number sequences and $x_n = a_n x_{n-1} + b_n x_{n-2}$ for $n \geq 3$, then

$$x_n = \sum_{j=1}^{f_n} ((x_1 - x_2)(\mathbf{f})_j + x_2) \left(\prod_{\substack{3 \leq k \leq n \\ (\mathbf{f}_k)_{j=1}}} b_k \right) \left(\prod_{\substack{3 \leq k \leq n \\ (\mathbf{f}_k + \mathbf{f}_{k+1})_j = 0}} a_k \right),$$

where f_n is n th Fibonacci number, $\mathbf{f} = (0, 1, 0, 0, \dots)$ is infinite Fibonacci word; \mathbf{f}_k is obtained from \mathbf{f} by replacing each entry of zero with f_{k-1} zeros and each entry of one with f_{k-2} ones.

Proof. Define vectors: $\mathbf{x}_1 = (x_1)$, $\mathbf{x}_2 = (x_2)$, $\mathbf{x}_n = (b_n \mathbf{x}_{n-2}) \cup (a_n \mathbf{x}_{n-1})$ for $n \geq 3$ and $\mathbf{y}_1 = \mathbf{x}_1$, $\mathbf{y}_n = \mathbf{y}_{n-1} \cup \mathbf{x}_n$, $\mathbf{p}_n = \mathbf{1}_{f_{n+1}-1} \cup (b_n \mathbf{1}_{f_{n-2}}) \cup (a_n \mathbf{1}_{f_{n-1}})$ for $n \geq 2$.

Let $\Lambda_k \mathbf{b} \stackrel{\text{def}}{=} \mathbf{b} \cup (\mathbf{b})_k^{|\mathbf{b}|}$. We have $\mathbf{y}_n = \mathbf{p}_n \Lambda_{f_{n-1}} \mathbf{y}_{n-1}$ for $n \geq 3$, from which it follows that $\mathbf{y}_n = \mathbf{y}_{2,n} \prod_{k=3}^n \mathbf{p}_{k,n}$, where $\mathbf{y}_{2,n} = \Lambda_{f_{n-1}} \Lambda_{f_{n-2}} \dots \Lambda_{f_2} \mathbf{y}_2$, $\mathbf{p}_{k,n} = \Lambda_{f_{n-1}} \Lambda_{f_{n-2}} \dots \Lambda_{f_k} \mathbf{p}_k$.

Partition $y_{2,n} = y'_1 \cup \dots \cup y'_n$ such that $|y'_i| = f_i$, then $y'_1 = x_1, y'_2 = x_2, y'_n = y'_{n-2} + y'_{n-1}$ for $n \geq 3$. Similarly partition $p_{k,n} = p'_{k,1} \cup \dots \cup p'_{k,n}$ such that $|p'_{k,i}| = f_i$, then $p'_{k,i} = 1_{f_i}$ for $1 \leq i \leq k-1, p'_{k,k} = (b_k 1_{f_{k-2}}) \cup (a_k 1_{f_{k-1}}), p'_{k,i} = p'_{k,i-2} \cup p'_{k,i-1}$ for $k+1 \leq i \leq n$.

Note that $(x_n)_- = (y'_n)_- \prod_{k=3}^n (p'_{k,n})_-$, where by $(a)_-$ we denote the vector composed of elements of a in reverse order. Now $(y'_n)_-$ and $(p'_{k,n})_-$ can be expressed in terms of infinite generalized Fibonacci words: $(y'_n)_- = (x_1 - x_2)(f_1)^{f_n} + x_2 1_{f_n}, (p'_{k,n})_- = (f_{k+1} + b_k f_k + a_k(1_{f_n} - f_{k+1} - f_k))_1^{f_n}$.

Finally using $x_n = |(x_n)_-|_1$ we get the result. □

The same sequence can be expressed with help of the Kronecker product.

Theorem 2. *If x_1, x_2 are arbitrary numbers, a_n, b_n are arbitrary number sequences and $x_n = a_n x_{n-1} + b_n x_{n-2}$ for $n \geq 3$, then*

$$x_n = \sum_{\substack{2^{n-2}+1 \leq j \leq 2^{n-1} \\ \vartheta(2^{n-1}-j+1)=1}} ((x_2 - x_1)[1, 1]_j + x_1) \prod_{\substack{0 \leq k \leq n-3 \\ [3 \cdot 2^k, 2^k]_j=1}} a_{k+3} \prod_{\substack{0 \leq k \leq n-3 \\ [2^k, 2^k]_j=0}} b_{k+3},$$

where $\vartheta(n) \stackrel{\text{def}}{=} \prod_{k=0}^{\infty} (1 - [3 \cdot 2^k, 2^k]_n)$.

Proof. Define vectors: $r_1 = (x_1, x_2), r_n = (1_2 \times r_{n-1})(h_n \times 1_{2^{n-2}})$ for $n \geq 2$, where $h_n = (0, 1, b_{n+1}, a_{n+1})$. It can be easily shown that $|(r_n)_{2^{n-1}+1}^{2^n}|_1 = x_{n+1}$. Solving the recurrence equation we get: $r_n = (1_{2^{n-1}} \times r_1) \prod_{k=2}^n (1_{2^{n-k}} \times h_k \times 1_{2^{k-2}})$. Taking into account that $h_k = (0, 1, 1, 1) (b_{k+1}, 1, b_{k+1}, 1) (1, 1, 1, a_{k+1})$ and doing some calculations we get the result. □

The following lemma allows us to generalize the result to the nonhomogeneous case.

Lemma 1. *If x_1 is arbitrary vector, b_n is 0,1-vector sequence, a_n and c_n are such that $|a_n| = |c_n| = |x_1| \prod_{i=2}^n |b_i|$; $x_n = a_n(b_n \times x_{n-1}) + c_n$ for $n \geq 2$, then*

$$x_n = (b_{n,2} \times x_1) \prod_{k=2}^n (b_{n,k+1} \times a_k) + \sum_{i=2}^n (b_{n,i+1} \times c_i) \prod_{k=i+1}^n (b_{n,k+1} \times a_k),$$

where $b_{n,k} = b_n \times b_{n-1} \times \dots \times b_k$, if $k \leq n$ and $b_{n,k} = 1_1$, if $k > n$.

The proof is straightforward.

Vectors r'_n for similar nonhomogeneous sequence $x'_n = a_n x'_{n-1} + b_n x'_{n-2} + c_n$ such that $|(r'_n)_{2^{n-1}+1}^{2^n}|_1 = x'_{n+1}$, are defined as follows: $r'_1 = (x_1, x_2), r'_n = (1_2 \times r'_{n-1})(h_n \times 1_{2^{n-2}}) + c_n(0_{2^{n-1}} \cup 1_1)$ for $n \geq 2$. To use the lemma we should, of course, do substitutions $a_n = h_n \times 1_{2^{n-2}}$ and $b_{n,k} = 1_{2^{n-k+1}}$.

Theorem 3. *If w_0 is arbitrary number, $a_{n,j}$ is arbitrary number sequence and $w_n = \sum_{j=0}^{n-1} a_{n,j} w_j$ for $n \in \mathbb{N}$, then*

$$\sum_{j=0}^n w_j = w_0 \sum_{v \in \mathbb{V}_n} \prod_{k=1}^{|v|} a_{(v)_k, (0_1 \cup v)_k}$$

where set \mathbb{V}_n consists of all vectors v such that $1 \leq (v)_{i-1} < (v)_i \leq n$ for $2 \leq i \leq |v|$.

Proof. Define vectors: $w_0 = (w_0), w_1 = (w_0, a_{1,0} w_0), w_n = w_{n-1} \cup (w_{n-1} q_n)$ for $n \geq 2$, where $(q_n)_1 = a_{n,0}, (q_n)_{2^{k-1}+1}^{2^k} = a_{n,k} 1_{2^{k-1}}$ for $1 \leq k \leq n-1$. From the recurrence equation it follows that if equality $|w_l|_1 = \sum_{j=0}^l w_j$ is true for $l = n-1 \geq 1$, then it is true for $l = n$; it is true for $l = 1$, so we conclude that it is true for any $l \in \mathbb{N}$. Solving the recurrence equation, we get: $w_n = w_0 \prod_{k=1}^n (1_{2^{n-k}} \times (1_{2^{k-1}} \cup q_k))$ for $n \in \mathbb{N}$. Noting that $q_k = (a_{k, \lceil \log_2 j \rceil})_{j=1}^{2^{k-1}}$, we have

$$|w_n|_1 = w_0 \sum_{j=1}^{2^n} \prod_{k=1}^n ([2^{k-1}, 2^{k-1}]_j (a_{k, \lceil \log_2(1+(j-1) \bmod 2^{k-1}) \rceil} - 1) + 1).$$

The quantity $[2^k, 2^k]_j$ equals the value of k th digit of number $(j - 1)$ in binary numeral system. If k_i is serial number of i th digit 1, then $\lceil \log_2(1 + (j - 1) \bmod 2^{k_i}) \rceil = 1 + k_{i-1}$. Therefore, $|w_n|_1 = w_0 \sum_j \prod_i a_{k_i+1, k_{i-1}+1}$, assuming $k_0 = -1$. Here $(k_i + 1)$ ranges over $v \in \mathbb{N}_n: k_i + 1 = (v)_i$. \square

In the same way vectors w'_n for nonhomogeneous sequence $w'_n = c_n + \sum_{j=0}^{n-1} a_{n,j} w'_j$ such that $|w'_1|_1 = \sum_{j=0}^1 w_j$, are defined as follows: $w'_0 = (w_0)$, $w'_1 = (w_0, c_1 + a_{1,0} w_0)$, $w'_n = w'_{n-1} \cup (w'_{n-1} q_n) + c_n (0_{2^n-1} \cup 1_1)$ for $n \geq 2$.

3. Quadratic map

It is well-known that in many cases iterations of a polynomial of degree 2 in the general case, i.e. solutions of quadratic map, can be expressed by iterations of a polynomial of degree 2 with one parameter.

Theorem 4. Let $p^{(0)}(x) = x$, $p^{(n)}(x) = p^{(n-1)}(p(x))$ (for $n \in \mathbb{N}$) be iterations of polynomial $p(x) = \lambda(x + 1)x$, then

$$p^{(n)}(x) = \sum_{k=1}^{2^n} x^k \sum_{i=(k-1)\omega_n}^{k\omega_n-1} \prod_{j=1}^n \lambda^{\mu_{i,j-1}} \binom{\mu_{i,j-1}}{\mu_{i,j} - \mu_{i,j-1}},$$

where $\omega_n = 2^{\frac{n(n-1)}{2}}$, $\mu_{i,j} = \lceil \frac{1+i \bmod \omega_{j+1}}{\omega_j} \rceil$.

Proof. Any polynomial $p^{(n)}(x)$ can be expressed as follows: $p^{(n)}(x) = \sum_{k=1}^{2^n} g_{n,k} x^k$, where $g_{n,k} = g_{n,k}(\lambda)$ are polynomials defined by equalities: $g_{0,1} = 1$, $g_{n,k} = \sum_{i=1}^{2^{n-1}} q_{k,i} g_{n-1,i}$ (for $1 \leq k \leq 2^n$), where $q_{k,i} = \lambda^i \binom{i}{k-i}$.

Let $p_0 = (1)$, $p_n = (1_{2^n} \times p_{n-1})(q_n \times 1_{\omega_{n-1}})$ for $n \in \mathbb{N}$, where

$$q_n = (q_{1,j})_{j=1}^{2^{n-1}} \cup (q_{2,j})_{j=1}^{2^{n-1}} \cup \dots \cup (q_{2^n,j})_{j=1}^{2^{n-1}} = \left(q_{\lceil i/2^{n-1} \rceil, 1+(i-1) \bmod 2^{n-1}} \right)_{i=1}^{2^{2^n-1}}.$$

Then $|(p_n)_{1+(k-1)\omega_n}^{k\omega_n}|_1 = g_{n,k}$. Solving the equation, we get:

$$p_n = \prod_{k=1}^n \left(1_{2^{\frac{(n+k+1)(n-k)}{2}}} \times q_k \times 1_{\omega_{k-1}} \right).$$

The last step is to check that $1_\infty \times q_k \times 1_{\omega_{k-1}} = \left(q_{\mu_{j,k}, \mu_{j,k-1}} \right)_{j=0}^\infty$. \square

Remark 1. Using generating polynomial $|q_k|_t = \lambda(1 + t^m)((\lambda t^{m+1} + \lambda t^{2m+1})^m - 1) / (\lambda t^{m+1} + \lambda t^{2m+1} - 1)$, where $m = 2^{k-1}$ and taking in account formula $|a \times b|_t = |a|_{|t|} |b|_t$ we can represent $p^{(n)}(x)$ as Hadamard's product of n functions. The polynomial $|q_k|_t$ can be derived using polynomials $\varphi_{k,m} = \sum_{j=1}^m \binom{j}{k-j} (\lambda t)^{j-1}$ as follows. Taking in account that $\varphi_{k,m} = \lambda t(\varphi_{k-1,m} + \varphi_{k-2,m}) - (\lambda t)^m \binom{m+1}{k-m-1}$ for $k \geq 3$ and $m \geq 1$, we can write

$$\begin{aligned} \lambda^{-1} |q_k|_t &= \sum_{k=1}^{2m} t^{(k-1)m} \varphi_{k,m} = 1 + t^m + \lambda t^{m+1} \sum_{k=1}^{2m-1} t^{(k-1)m} \varphi_{k,m} + \lambda t^{2m+1} \sum_{k=1}^{2m-2} t^{(k-1)m} \varphi_{k,m} - \lambda^m \sum_{k=2}^{2m} t^{km} \binom{m+1}{k-m-1} \\ &= 1 + t^m + \lambda t^{m+1} (\lambda^{-1} |q_k|_t - t^{m(2m-1)} \varphi_{2m,m}) + \lambda t^{2m+1} (\lambda^{-1} |q_k|_t - t^{m(2m-1)} \varphi_{2m,m} - t^{m(2m-2)} \varphi_{2m-1,m}) \\ &\quad - \lambda^m (t^{m(m+1)} (1 + t^m)^{m+1} - (m + t^m + 1) t^{2m^2+m}). \end{aligned}$$

From here taking in account $\varphi_{2m,m} = (\lambda t)^{m-1}$ and $\varphi_{2m-1,m} = m(\lambda t)^{m-1}$ we immediately get $|q_k|_t$.

Let's consider another episode. Let $s^{(0)}(x) = x$, $s^{(n)}(x) = s^{(n-1)}(s(x))$ (for $n \in \mathbb{N}$) be iterations of polynomial $s(x) = s_\lambda(x) = \lambda(x^2 - 1) + 1$ and $\lambda \neq 0$. Define vectors: $s_1 = (x - 1, 1)$, $s_n = \lambda s_{n-1}^{(2)} - (\lambda - 1)(0_{2^{n-1}-1} \cup 1)$ for $n \geq 2$, where triangular brackets indicate Kronecker degree. Obviously, $|s_n|_1 = s(|s_{n-1}|_1) = s^{(n)}(x)$.

Solving the equation, we get:

$$s_n = \lambda s_{n-1}^{(2)} \mathbf{1}_{n-1} = \lambda^{2^{n-1}-1} s_1^{(2^{n-1})} r_{\lambda, n-1} = \lambda^{2^{n-1}-1} s_1^{(2^{n-1})} (r^-)_1^{2^{n-1}} \tag{1}$$

where $\mathbf{1}_n = \mathbf{1}_{2^{2^n-1} \cup (\lambda^{-1})}$, $r_{\lambda, n} = \prod_{i=1}^n \mathbf{1}_i^{(2^{n-i})}$, $r^- = \prod_{i=1}^\infty \mathbf{1}_i^{(\infty)}$.

Therefore, we have

$$s_\lambda^{(n)}(x) = \lambda^{2^{n-1}-1} \sum_{j=1}^{2^n} (x-1)^{(h_n)_j} \lambda^{-(\log_2 r_{2, n-1})_j} \tag{2}$$

where $h_n = \log_2 (2, 1)^{(2^{n-1})}$. And it can be easily shown that

$$(h_n)_j = 2^{n-1} - \sum_{k=0}^\infty [2^k, 2^k]_j \text{ and } (\log_2 r_{2, n})_j = \sum_{k,i=0}^\infty [2^{2^i k} (2^{2^i} - 1), 2^{2^i k}]_j$$

by using simple formula

$$(\log_2 (1_{m-1} \cup (2)))^{(n)}_j = \sum_{k=0}^{n-1} [m^k (m-1), m^k]_j \text{ (for } j \leq m^n \text{)}.$$

Substituting 2 for x in (2), we have

$$s_\lambda^{(n)}(2) = \lambda^{2^{n-1}-1} \sum_{k=1}^{2^{n-1}} \kappa_{n,k} \lambda^{-k},$$

where $\kappa_{n,k}$ denotes the number of elements in $\log_2 r_{2, n-1}$ that equal to k . This function can be defined recursively as follows:

$$\kappa_{n,0} = 3^{2^{n-1}}, \kappa_{1,k} = \delta_{k,1}, \kappa_{n,k} = \delta_{k,2^{n-1}} - \delta_{k,2^{n-2}} + \sum_{i=0}^k \kappa_{n-1,k-i} \kappa_{n-1,i}.$$

Evaluating $\kappa_{n,k}$ we have:

$$\kappa_{n,1} = 2^{n-1} 3^{2^{n-1}-1}, \kappa_{n,2} = -\kappa_{n,1} \left(\frac{1}{12} - \sum_{i=0}^{n-1} \kappa_{i+1,1} (\kappa_{i,1}^2 + \delta_{k,2^{n-1}} - \delta_{k,2^{n-2}}) \right)$$

and so on.

Remark 2. Replacing 2^{n-1} by n in the last expression of (1), we get new sequence of vectors: $s'_n = \lambda^{n-1} s_1^{(n)}(x)_1^{2^n}$. Let $f_n(x) = |s'_n|_1$. Conjecture:

$$f_n(x) = \begin{cases} f_{n/2}(s(x)), & \text{if } n \text{ is even} \\ \lambda x f_{n-1}(x), & \text{if } n \text{ is odd} \\ x, & \text{if } n = 1. \end{cases}$$

Conflicts of Interest: "The author declares no conflict of interest."

References

[1] Popenda, J. (1987). One expression for the solutions of second order difference equations. *Proceedings of the American Mathematical Society*, 100(1), 87-93.
 [2] Kittappa, R. K. (1993). A representation of the solution of the nth order linear difference equation with variable coefficients. *Linear algebra and its applications*, 193, 211-222.

- [3] Mallik, R. K. (2000). On the solution of a linear homogeneous difference equation with variable coefficients. *SIAM Journal on Mathematical Analysis*, 31(2), 375-385.
- [4] Allouche, J. P., & Shallit, J. (2003). *Automatic sequences: theory, applications, generalizations*. Cambridge university press.
- [5] Lando, S. K. (2003). *Lectures on generating functions* (Vol. 23). American Mathematical Society.
- [6] Voevodin, V., & Kuznetsov Y. (1984). *Matrices and calculations*. Nauka (in Russian).
- [7] Fomin, S. (1975). *Number systems*. The University of Chicago Press.



© 2020 by the author; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (<http://creativecommons.org/licenses/by/4.0/>).