Open Journal of
Discrete Applied Mathematics

PSR Press

Article

Minimal graphs for hamiltonian extension

Christophe Picouleau

CEDRIC laboratory, Conservatoire National des Arts et Métiers, Paris, France.; christophe.picouleau@cnam.fr

Received: 16 January 2020; Accepted: 2 March 2020; Published: 8 March 2020.

Abstract: For every $n \geq 3$, we determine the minimum number of edges of graph with n vertices such that for any non edge xy there exits a hamiltonian cycle containing xy.

Keywords: 2-factor, hamiltonian cycle, hamiltonian path.

MSC: 26B25, 26A33, 26A51, 33E12.

---

1. Introduction

For all graph theoretical terms and notations not defined here the reader is referred to [1]. We only consider simple finite loopless undirected graphs. For a graph $G = (V, E)$ with $|V| = n$ vertices, an edge is a pair of two connected vertices x, y, we denote it by $xy, xy \in E$; when two vertices x, y are not connected this pair form the *non-edge* $xy, xy \notin E$. In G a 2-factor is a subset of edges $F \subset E$ such that every vertex is incident to exactly two edges of F. Since G is finite a 2-factor consists of a collection of vertex disjoint cycles spanning the vertex set V. When the collection consists of an unique cycle the 2-factor is connected, so it is a hamiltonian cycle.

We intend to determine, for any integer $n \geq 3$, a graph $G = (V, E), n = |V|$ with a minimum number of edges such that for every non-edge xy it is always possible to include the non-edge xy into a connected 2-factor, i.e., the graph $G_{xy} = (V, E \cup \{xy\})$ has a hamiltonian cycle $H, xy \in H$. In other words for any non-edge xy of G there exits a hamiltonian path between x and y.

This problem is related to the minimal 2-factor extension studied in [2] in which the 2-factors are not necessary connected. It is also related to the problem of finding minimal graphs for non-edge extensions in the case of perfect matchings (1-factors) studied in [3]. Another problem of hamiltonian extension can be found in [4].

Definition 1. Let $G = (V, E)$ be a graph and $xy \notin E$ an non-edge. If $G_{xy} = (V, E \cup \{xy\})$ has a hamitonian cycle that contains xy we shall say that xy has been *extended* (to a connected 2-factor, to an hamiltonian cycle).

Definition 2. A graph $G = (V, E)$ is *connected 2-factor expandable* or *hamiltonian expandable* (shortly *expandable*) if every non-edge $xy \notin E$ can be extended.

Definition 3. An expandable graph $G = (V, E)$ with $|V| = n$ and a minimum number of edges is a *minimum expandable graph*. The size $|E|$ of its edge set is denoted by $Exp_h(n)$.

The case where the 2-factor is not constrained to be hamiltonian is studied in [2]. In this context $Exp_2(n)$ denotes the size of a *minimum expandable graph* with n vertices. It follows that $Exp_h(n) \geq Exp_2(n)$.

We use the following notations. For $G = (V, E)$, $N(v)$ is the set of neighbors of a vertex v, $\delta(G)$ is the minimum degree of a vertex. A vertex with exactly k neighbors is a k-vertex. When $P = v_i, \ldots, v_j$ is a sequence of vertices that corresponds to a path in G, we denote by $\bar{P} = v_j, \ldots, v_i$ its mirror sequence (both sequences correspond to the same path).

We state our result.

Theorem 1. *The minimum size of a connected 2-factor expandable graph is:*

$$Exp_h(3) = 2, Exp_h(4) = 4, Exp_h(5) = 6; Exp_h(n) = \lceil \frac{3}{2}n \rceil, n \geq 6$$

Proof. For $n \geq 3$ we have $Exp_h(n) \geq Exp_2(n)$.

In [2] it is proved that the three graphs given by Figure 1 are minimum for 2-factor extension. They are also minimum expandable for connected 2-factor extension.

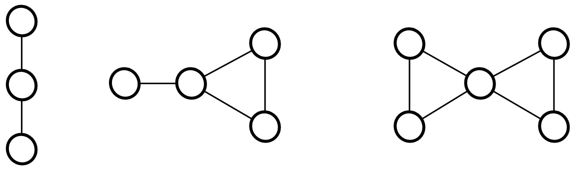

Figure 1. P_3, the paw, the butterfly.

Now let $n \geq 6$. From [2] we know the following when G a minimum expandable graph for the 2-factor extension:

- G is connected;
- if $\delta(G) = 1$ then $Exp_2(n) \geq \frac{3}{2}n$;
- for $n \geq 7$, if u, v are two 2-vertices such that $N(u) \cap N(v) \neq \emptyset$ then $Exp_2(n) \geq \frac{3}{2}n$;

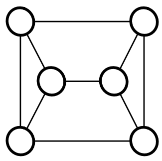

Figure 2. A minimum hamiltonian expandable graph with 6 vertices.

The graph given by Figure 2 is minimum for 2-factor extension (see [2]). One can check that it is expandable for connected 2-factor extension. So we have $Exp_h(6) = 9 = \frac{3}{2}n$.

Suppose that G is a minimum expandable graph with $n \geq 7$ and $\delta(G) = 2$. Let $v \in V$ with $d(v) = 2$, $N(v) = \{u_1, u_2\}$. If $u_1u_2 \notin E$ then u_1u_2 cannot be expanded into a hamiltonian cycle. So $u_1u_2 \in E$. If $d(u_1) = 2$ then $u_2 \in N(u_1) \cap N(v)$ and $Exp_h(n) \geq \frac{3}{2}n$. So from now one we may assume $d(u_1), d(u_2) \geq 3$. Suppose that $d(u_1) = d(u_2) = 3$. Let $N(u_1) = \{v, u_2, v_1\}, N(u_2) = \{v, u_1, v_2\}$. If $v_1 \neq v_2$ then u_1v_2 is not expandable. If $v_1 = v_2$ then vv_1 is not expandable. From now we can suppose that $d(u_1) \geq 3, d(u_2) \geq 4$. Moreover v is the unique 2-vertex in $N(u_2)$. It follows that every 2-vertex $u \in V$ can be matched with a distinct vertex u_2 with $d(u_2) \geq 4$. Then $\Sigma_{v \in V} d(v) \geq 3n$ and thus $m \geq \frac{3}{2}n$.

When $\delta(G) \geq 3$ we have $\Sigma_{v \in V} d(v) \geq 3n$. Thus for any expandable graph we have $|E| = m \geq \frac{3}{2}n, n \geq 7$.

For any even integer $n \geq 8$ we define the graph $G_n = (V, E)$ as follows. Let $n = 2p$, $V = A \cup B$ where $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_p\}$. A (resp. B) induces the cycle $C_A = (A, E_A)$ with $E_A = \{a_1a_2, a_2a_3, \ldots, a_pa_1\}$ (resp. $C_B = (B, E_B)$ with $E_B = \{b_1b_2, b_2b_3, \ldots, b_pb_1\}$. Now $E = E_A \cup E_B \cup E_C$ with $E_C = \{a_2b_2, a_3b_3, \ldots, a_{p-1}b_{p-1}, a_1b_p, a_pb_1\}$. Note that G_n is cubic so $m = \frac{3}{2}n$. (see G_{10} in Fig. 3)

We show that G_n is expandable. First we consider a non-edge $a_ia_j, p \geq j > i \geq 1$. Note that the case of a non-edge b_ib_j is analogous. We have $j \geq i + 2$ and since $a_1a_p \in E$ from symmetry we can suppose that $j < p$. Let $P = a_j, a_{j-1}, \ldots, a_{i+1}, b_{i+1}, b_{i+2}, \ldots, b_{j+1}, a_{j+1}, a_{j+2}, b_{j+2}, \ldots, c_j$ where c_j is either a_p or b_p and let $Q = a_i, b_i, b_{i-1}, a_{i-1}, \ldots, c_i$ where c_i is either a_1 or b_1. From P and Q one can obtain an hamiltonian cycle containing a_ib_j whatever c_i and c_j are.

Now we consider a non-edge a_ib_j. Without loss of generality we assume $j \geq i$. Suppose first that $j = i$, so either $i = 1$ or $i = p$. Without loss of generality we assume $i = j = 1$: $a_1, b_p, b_{p-1}, \ldots, b_2, a_2, a_3, \ldots, a_p, b_1, a_1$ is an

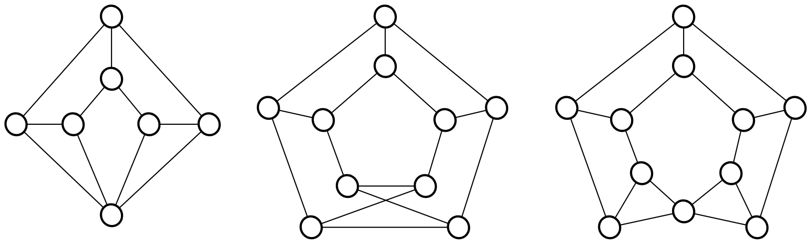

Figure 3. The graphs G_7, G_{10}, G_{11}, from the left to the right.

hamiltonian cycle. Now assume that $j > i$: Let $P_j = b_j, b_{j-1}, \dots, b_{i+1}, a_{i+1}, a_{i+2}, \dots, a_{j+1}, b_{j+1}, b_{j+2}, a_{j+2}, \dots, c_p$ where either $c_p = a_p$ or $c_p = b_p$, $P_i = a_i, b_i, b_{i-1}, a_{i-1}, a_{i-2}, \dots, c_1$ where either $c_1 = a_1$ or $c_1 = b_1$. If $c_p = a_p$ and $c_1 = a_1$ then P_j, b_1, b_p, P_i, a_j is an hamiltonian cycle. If $c_p = a_p$ and $c_1 = b_1$ then P_j, a_1, b_p, P_i, a_j is an hamiltonian cycle. The two other cases are symmetric.

For any odd integer $n = 2p + 1 \geq 7$ we define the graph $G_n = (V, E)$ as follows. We set $V = A \cup B \cup \{v_n\}$ where $A = \{a_1, \dots, a_p\}$ and $B = \{b_1, \dots, b_p\}$. $A \cup \{v_n\}$ (resp. $B \cup \{v_n\}$) induces the cycle $C_A = (A \cup \{v_n\}, E_A)$ with $E_A = \{a_1a_2, a_2a_3, \dots, a_pv_n, v_na_1\}$ (resp. $C_B = (B \cup \{v_n\}, E_B)$ with $E_B = \{b_1b_2, b_2b_3, \dots, b_pv_n, v_nb_1\}$. Now $E = E_A \cup E_B \cup E_C$ with $E_C = \{a_ib_i | 1 \leq i \leq p\} \cup \{a_1v_n, b_1v_n, a_pv_n, b_pv_n\}$. Note that $m = \lceil \frac{3}{2}n \rceil$. (see G_7 and G_{11} in Figure 3)

We show that G_n is expandable. First, we consider a non-edge $a_ia_j, p \geq j > i \geq 1$ (the case of a non-edge b_ib_j is analogous). $a_i, a_{i+1}, \dots, a_{j-1}, b_{j-1}, b_{j-2}, b_{j-3}, \dots, b_i, b_{i-1}, a_{i-1}, a_{i-2}, b_{i-2}, \dots, v_n, c_p, d_p, d_{p-1}, c_{p-1}, \dots, c_j, d_j$, where $d_j = a_j$ and for any $k, j \leq k \leq p$, the ordered pairs c_k, d_k correspond to either a_k, b_k or b_k, a_k, is an hamiltonian cycle. Second, let a non-edge $a_ib_j, p \geq j > i \geq 1$. We use the same construction as above taking $d_j = b_j$. □

Acknowledgments: The author express its gratitude to Dominique de Werra for its constructive comments and remarks, which helped to improve the writing of this paper.

Conflicts of Interest: "The author declare no conflict of interest."

References

[1] Bondy, J.A., Murty, U. S. R. (2008). Graph Theory. Springer.

[2] Costa, M. C., de Werra, D., & Picouleau, C. (2020). Minimal graphs for 2-factor extension. *Discrete Applied Mathematics*, accepted for publication.

[3] Costa, M. C., de Werra, D., & Picouleau, C. (2018). Minimal graphs for matching extensions. *Discrete Applied Mathematics, 234*, 47-55.

[4] Zhang, P. (2016). Hamiltonian Extension. In *Graph Theory* (pp. 17-30). Springer, Cham.