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A NEW THIRD-ORDER ITERATION METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this paper, we establish a two step third-order iteration method for solving nonlinear equations. The efficiency index of the method is 1.442 which is greater than Newton-Raphson method. It is important to note that our method is performing very well in comparison to fixed point method and the method discussed by Kang *et al.* (Abstract and applied analysis; volume 2013, Article ID 487060).

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1. Introduction

Solving equations in one variable is the most discussed problem in numerical analysis. There are several numerical techniques for solving nonlinear equations (see for example [1, 2, 3, 4, 5, 6, 7, 8] and the references there in). For a given function f, we have to find at least one solution to the equation $f(x) = 0$. Note that, priory, we do not put any restrictions on the function f. In order to check whether a given solution is true or not, we need to be able to evaluate the function, that is, $f(\alpha) = 0$. In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of "good behavior". The more we assume, the more potential we have to develop fast algorithms for finding the root. At the same time, more assumptions will reduce the number of function classes which will our assumptions. This is a fundamental paradigm in numerical analysis.

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We present a new iteration method that approximates the root of a nonlinear equation in one variable using the value of the function and its derivative. Our method converges to the root cubically. In this study, we suggest an improvement to the iteration of Kang iteration method at the expense of one additional first derivative evaluation. It is shown that the suggested method converges to the root, and the order of convergence is at least three in a neighborhood of the root, whenever the first and higher order derivatives of the function exist in a neighborhood of the root. This means that our method approximately triples the number of significant digits after an iteration. Numerical examples support this theory, and the computational order of convergence is even more than three for certain functions.
We know that fixed point iteration method [9] is the fundamental algorithm for solving nonlinear equations in one variable. In this method equation is rewritten as

$$x = g(x), \tag{1}$$

where, the following are true:

(1) $\exists\ [a,b]$ such that $g(x) \in [a,b]$ for all $x \in [a,b]$,
(2) $\exists\ [a,b]$ such that $|g'(x)| \leq L < 1$ for all $x \in [a,b]$.

Definition 1.1. Suppose, $\{x_n\} \to \alpha$, with the property that,

$$\lim_{n\to\infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^q} \right| = D,$$

where, $D \in \mathbb{R}^+$ and $q \in \mathbb{Z}$, then D is called the constant of convergence and q is called the order of convergence.

Lemma 1.2. [2]
Let $g \in C^p[a,b]$. If $g^{(k)}(x) = 0$ for $k = 1, 2, \ldots, p-1$ and $g^{(p)}(x) \neq 0$, then the sequence $\{x_n\}$ is of order p.

Algorithm 1.3. [10]
For a given x_0, Kang *et. al.* gave the approximate solution x_{n+1} by an iteration scheme as follows.

$$x_{n+1} = \frac{g\left(x_n\right) - x_n g'(x_n)}{1 - g'(x_n)}$$

where, $g'(x_n) \neq 1$. This scheme has convergence of order 2.

2. New Iteration Method

In the fixed-point iteration method, for some $x \in \mathbb{R}$, if $f\left(x\right) = 0$ then the nonlinear equation can be converted to,

$$x = g(x).$$

Let α be the root of $f\left(x\right) = 0$. We can write functional equation of algorithm 1.3 as

$$H_g(x) = \frac{g\left(x\right) - xg'(x)}{1 - g'(x)}, \qquad g'(x) \neq 1$$

or,

$$H_g(x) = x - \frac{x - g(x)}{1 - g'(x)}.$$

To get higher order convergence, we introduce $h(x)$ in above, as follows

$$H_h(x) = x - \frac{x - g(x)}{1 - g'(x + (x - g(x))h(x))}. \tag{2}$$

Then $H_h(\alpha) = \alpha$ and $H'_h(\alpha) = 0$. In order to make (2) efficient, we shall choose $h(x)$ such that $H''_h(\alpha) = 0$. By using Mathematica we have,

$$H''_h(\alpha) = \frac{(1 - 2h(\alpha)(-1 + g'(\alpha)))g''(\alpha)}{-1 + g'(\alpha)}.$$

Then, $H''_h(\alpha) = 0$ gives,

$$h(a) = \frac{-1}{2(1 - g'(\alpha))}.$$

So, if we take $h\,(x) = \frac{-1}{2(1-g'(x))}$ and substitute it in (2), we get

$$H_h(x) = x - \frac{x - g(x)}{1 - g'(x - \frac{(x-g(x))}{2(1-g'(x))})}.$$

This formulation allows us to suggest the following two step iteration method for solving nonlinear equation, for a given x_0.

Algorithm 2.1.

$$x_{n+1} = x_n - \frac{x_n - g(x_n)}{1 - g'(y_n)}, \qquad n = 0, 1, 2, \dots \tag{3}$$

$$y_n = x_n - \frac{x_n - g(x_n)}{2(1 - g'(x_n))}, \qquad g'(x_n) \neq 1 \tag{4}$$

3. Convergence Analysis of Algorithm 2.1

Theorem 3.1. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a nonlinear function on an open interval D, such that $f\,(x) = 0$ (or equivalently $x = g\,(x)$), has a simple root $\alpha \in D$. Here, $g : D \subset \mathbb{R} \to \mathbb{R}$, is sufficiently smooth in the neighborhood of the root α. Then, the order of convergence of algorithm 2.1 is at least 3, where $c_k = \frac{g^{(k)}(\alpha)}{k!(1-g'(\alpha))}$, $k = 2, 3, \dots$.*

Proof. As $g\,(\alpha) = \alpha$, let $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$. By Taylor's expansion, we have,

$$g(x_n) = g\,(\alpha) + e_n g'(\alpha) + \frac{e_n^2}{2!} g''(\alpha) + \frac{e_n^3}{3!} g'''(\alpha) + O(e_n^4).$$

This implies that,

$$x_n - g(x_n) = (1 - g'(\alpha))e_n - c_2 e_n^2 - c_3 e_n^3 + O(e_n^4). \tag{5}$$

Similarly,

$$1 - g'(x_n) = (1 - g'(\alpha))1 - 2e_nc_2 - 3e_n^2c_3 - 4e_n^3c_4 + O(e_n^4).$$

Substituting these values in (4) we obtain

$$\begin{aligned} y_n &= x_n - \frac{(1 - g'(\alpha))e_n - c_2e_n^2 - c_3e_n^3 + O(e_n^4)}{2(1 - g'(\alpha))1 - 2e_nc_2 - 3e_n^2c_3 - 4e_n^3c_4 + O(e_n^4)} \\ &= \alpha + e_n - \frac{1}{2}\frac{e_n - c_2e_n^2 - c_3e_n^3 + O(e_n^4)}{1 - (2e_nc_2 + 3e_n^2c_3 + 4e_n^3c_2^2 + O(e_n^4))}. \end{aligned}$$

Using the series expansion above, we get

$$\begin{aligned} y_n &= \alpha + e_n - \\ &\quad \frac{1}{2}e_n - c_2e_n^2 - c_3e_n^3 + O(e_n^4)1 + (2e_nc_2 + 3e_n^2c_3 + 4e_n^3c_2^2 + O(e_n^4)) + \ldots \\ &= \alpha + \frac{1}{2}e_n - c_2e_n^2 - 2(c_2^2 + c_3)e_n^3 + O(e_n^4). \end{aligned}$$

This implies that,

$$\begin{aligned} g'(y_n) &= g'\alpha + \frac{1}{2}e_n - c_2e_n^2 - 2(c_2^2 + c_3)e_n^3 + O(e_n^4) \\ &= g'(\alpha) + \left[\frac{1}{2}(e_n - c_2e_n^2 - 2(c_2^2 + c_3)e_n^3 + O(e_n^4))\right] g''(\alpha) \\ &\quad + \frac{1}{2}\left[\frac{1}{2}(e_n - c_2e_n^2 - 2(c_2^2 + c_3)e_n^3 + O(e_n^4))\right]^2 g'''(\alpha) + \ldots. \end{aligned}$$

Thus, we get,

$$\begin{aligned} &1 - g'(y_n) \\ = \; &1 - g'(\alpha)\left[1 - c_2e_n + (c_2^2 - \frac{3}{4}c_3)e_n^2 + (2c_2^3 + 2c_2c_3 - c_4)e_n^3 + O(e_n^4)\right]. \end{aligned} \quad (6)$$

Using (5) and (6) in (4), we have,

$$e_{n+1} = e_n - \frac{e_n - c_2e_n^2 - c_3e_n^3 + O(e_n^4)}{1 - c_2e_n - (c_2^2 - \frac{3}{4}c_3)e_n^2 + O(e_n^3)}.$$

Using series expansion again, we get,

$$\begin{aligned} e_{n+1} &= e_n - e_n - c_2e_n^2 - c_3e_n^3 + O(e_n^4)\left[1 + c_2e_n - (c_2^2 - \frac{3}{4}c_3)e_n^2 + O(e_n^3)\right. \\ &\quad \left. - c_2e_n - (c_2^2 - \frac{3}{4}c_3)e_n^2 + O(e_n^3)^2 + \ldots\right] \\ &= 3c_2^2 + \frac{1}{4}c_3e_n^3 + O(e_n^4) \end{aligned}$$

□

4. Comparison

Comparison of Fixed Point Method (FPM), Kang Iteration Method (KIM) and our new iteration method (NIM), is shown in the following table, root corrected up to seventeen decimal places.

Example 4.1. Let $f(x) = x^3 - 23x - 135$, take $g(x) = 23 + \frac{135}{x}$, it can be observed from Table 1 that NIM in this paper is faster than FPM and KIM.

Table 1. Comparison of FPM, NIM, KIM

Method	N	N_f	x_0	x_{n+1}	$f(x_{n+1})$
FPM	24	24	2	$2.420536e-15$	27.84778272427181476
KIM	7	14	2	$2.781849e-20$	27.84778272427181484
NIM	4	12	2	$2.647181e-16$	27.84778272427181483

Example 4.2. Let $f(x) = x - \cos x$, take $g(x) = \cos x$, it can be observed from Table 2 that NIM in this paper is faster than FPM and KIM.

Table 2. Comparison of FPM, NIM, KIM

Method	N	N_f	x_0	x_{n+1}	$f(x_{n+1})$
FPM	91	91	6	$1.376330e-16$	0.73908513321516064
KIM	10	20	6	$1.083243e-29$	0.73908513321516064
NIM	5	15	6	$1.809632e-36$	0.73908513321516064

Example 4.3. Let $f(x) = x^3 + 4x^2 + 8x + 8$, take $g(x) = -1 - 1/2x^2 - 1/8x3$, it can be observed from Table 3 that NIM in this paper is faster than FPM and KIM.

Table 3. Comparison of FPM, NIM, KIM

Method	N	N_f	x_0	x_{n+1}	$f(x_{n+1})$
FPM	50	50	-1.7	$1.416263e-15$	-1.99999999999999965
KIM	5	10	-1.7	$7.836283e-27$	-2.00000000000000000
NIM	3	9	-1.7	$4.983507e-25$	-2.00000000000000000

Example 4.4. Let $f(x) = \ln(x-2) + x$, take $g(x) = 2 + e^{-x}$, it can be observed from Table 4 that NIM in this paper is faster than FPM and KIM.

Example 4.5. Let $f(x) = x^2 \sin x - \cos x$, take $g(x) = \sqrt{\frac{1}{\tan x}}$, it can be observed from Table 5 that NIM in this paper is faster than FPM and KIM.

Example 4.6. Let $f(x) = x^2 - 5x - 16$, take $g(x) = 5 + \frac{16}{x}$ it can be observed from Table 6 that NIM in this paper is faster than FPM and KIM.

Table 4. Comparison of FPM, NIM, KIM

Method	N	N_f	x_0	x_{n+1}	$f(x_{n+1})$
FPM	18	18	0.1	$1.163802e-15$	2.12002823898764110
KIM	5	10	0.1	$5.972968e-22$	2.12002823898764123
NIM	3	9	0.1	$8.594812e-22$	2.12002823898764123

Table 5. Comparison of FPM, NIM, KIM

Method	N	N_f	x_0	x_{n+1}	$f(x_{n+1})$
FPM	417	247	2	$2.668900e-16$	0.89520604538423175
KIM	7	14	2	$3.195785e-31$	0.89520604538423175
NIM	4	12	2	$1.746045e-23$	0.89520604538423175

Table 6. Comparison of FPM, NIM, KIM

Method	N	N_f	x_0	x_{n+1}	$f(x_{n+1})$
FPM	34	34	1	$6.417745e-16$	7.21699056602830184
KIM	7	14	1	$2.984439e-27$	7.21699056602830191
NIM	4	12	1	$2.797394e-26$	7.21699056602830191

5. Conclusions

A new third order iteration method for solving nonlinear equations has been introduced. By using some examples, performance of NIM is also discussed. Its performance is much better, in comparison to the fixed point method and the method presented in [10].

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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